

The Vapnik-Chervonenkis dimension of norms on \mathbb{R}^d

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Abstract

The Vapnik-Chervonenkis dimension of a collection of subsets of a set is an important combinatorial parameter in machine learning. In this paper we show that the VC dimension of the family of d -dimensional cubes in \mathbb{R}^d (that is, the closed balls according to the ℓ^∞ norm) is $\lfloor (3d+1)/2 \rfloor$. We also prove that the VC dimension of certain families of convex sets in \mathbb{R}^2 (including the balls of all norms) is at most 3, and that there is a norm on \mathbb{R}^d for every $d \geq 3$ the collection of whose balls has infinite VC dimension.

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1. Introduction

The notion of the Vapnik-Chervonenkis (VC) dimension of a collection of subsets of a space Ω has been extensively studied and used in the context of machine learning (see, e.g., [1]). A natural question to ask is what the VC dimension of collections of balls with regard to various norms on \mathbb{R}^d is, given their prominence in different learning rules. By the VC dimension of a norm, for short, we will mean the VC dimension of the collection of all closed (equivalently, open) balls with regard to this norm. While the VC dimension of the standard Euclidean norm in \mathbb{R}^d is known to be $d+1$ [2], the VC dimension of ℓ^p norms, or norms in general, remains unknown.

In this paper we show that the VC dimension of the ℓ^∞ norm on \mathbb{R}^d is $\lfloor (3d+1)/2 \rfloor$, give an upper bound of 3 on the VC dimension of any norm on \mathbb{R}^2 and exhibit norms on \mathbb{R}^d for dimension 3 and higher that have infinite

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VC dimension. A remaining question is the VC dimension of other ℓ^p norms. While the VC dimension of such norms for p sufficiently large should be at least that of the ℓ^∞ norm, the precise values of the VC dimension and its behaviour as p changes is not known.

To define the VC dimension, let Ω be a set and \mathcal{C} be a collection of subsets. If S is a finite subset of Ω and A is a subset of S , then we say that \mathcal{C} *carves out* A from S if there is a set C in \mathcal{C} such that $S \cap C = A$. The set S is *shattered* by \mathcal{C} if it carves out every subset of S . With these definitions, we have the following.

Definition 1. The VC dimension of a collection \mathcal{C} in Ω is the supremum of the cardinalities of finite sets that are shattered by \mathcal{C} .

When we refer to the VC dimension of a norm, we mean the VC dimension of the set of closed balls according to that norm. Hence the VC dimension of the supremum norm on \mathbb{R}^d , denoted $\ell^\infty(d)$, is the VC dimension of the set of closed cubes in that space. These are exactly the sets of the form

$$[a_1, b_1] \times \cdots \times [a_d, b_d]$$

where $a_1, b_1, \dots, a_d, b_d$ are real numbers, $a_i \leq b_i$ for all i and $|b_i - a_i|$ is constant. We adopt the convention that cubes of diameter 0 (singletons) are included. The inclusion or exclusion of these does not affect the eventual VC dimension, as cubes may be made small enough to carve out any singleton, and if a cube contains two distinct points then it must have positive diameter. On the other hand, singletons will be excluded in sections 5 and 6. In both cases the decision is one of convenience, and does not affect the VC dimension.

2. The VC dimension of cubes when $d = 1$

The general method of proving that the VC dimension of a collection is a certain value is to proceed in two stages. In one stage an upper bound on the VC dimension, say k , is proven by showing that no set of cardinality $k + 1$ can be shattered by the collection. In the other a specific subset of cardinality k is given that may be shattered by the collection. This process is illustrated in the following standard proof that the VC dimension of cubes in \mathbb{R}^1 is 2, a number that agrees with the claim that the VC dimension of cubes in \mathbb{R}^d is $\lfloor (3d + 1)/2 \rfloor$.

Note that in this case the collection of closed cubes is exactly the set of closed intervals $\{[a, b] | a, b \in \mathbb{R}, a \leq b\}$. To establish the upper bound, let $S = \{a, b, c\}$ be a set of three elements in \mathbb{R} and assume that $a < b < c$. Then the subset $\{a, c\}$ of S cannot be carved out by a closed interval; any such interval must contain the points between a and c , and hence must contain b . So S cannot be shattered by closed intervals.

To establish the lower bound, consider the set $\{0, 1\}$ in \mathbb{R} . The subsets

$$\emptyset, \{0\}, \{1\}, \{0, 1\}$$

of this set may be carved out by the intervals

$$[3, 4], [0, 1/2], [1/2, 1], [0, 1]$$

respectively. So the set may be shattered by closed intervals. With the lower bound of 2 established, we have that the VC dimension of cubes in \mathbb{R}^1 is exactly 2.

3. Establishing an upper bound on the VC dimension

To establish that $VC(\ell^\infty(d))$ is at most $\lfloor (3d+1)/2 \rfloor$, let n be an integer at least zero and S be a subset of \mathbb{R}^d with $d+n$ points that may be shattered by \mathcal{C} so that we may derive a bound on n . We proceed by considering the points of S that attain the minimum and maximum values on each coordinate axis. To that end, for each integer i between 1 and d let l^i and u^i be points in S such that $l^i = \min\{s_i : s \in S\}$ and $u^i = \max\{s_i : s \in S\}$. Thus we have a list of pairs $(l^1, u^1), \dots, (l^d, u^d)$ of extrema of S .

First we note that every point of S must appear at least once in this list. Suppose that a point a in S does not, and let C be a cube that carves out the subset $S \setminus \{a\}$, say $C = C_1 \times \dots \times C_d$. Then for every i the points l^i and u^i lie in C . However $l^i \leq a_i \leq u^i$ and C_i is an interval, so a_i must be an element of C_i , and hence $a \in C$, a contradiction. So every point of S must appear at least once in the list of extrema.

Now, let k be the number of points that appear precisely once in this list of extrema. These k points are distributed over the $2d$ entries of the list, leaving $2d - k$ positions and $d + n - k$ points that must appear at least twice. Hence $2d - k \geq 2(d + n - k)$, implying that $k \geq 2n$.

Finally, to establish the upper bound on the VC dimension assume towards a contradiction that $k \geq d + 2$. By the pigeonhole principle there must

be distinct integers i, j between 1 and d such that l^i, u^i, l^j, u^j appear precisely once in the list of extrema. From our assumptions there is a cube C that carves out $S \setminus \{l^i, u^i\}$. Note that for $k \neq i$, by definition $l_k^i, u_k^i \in [l_k^k, u_k^k]$. The values l_k^k and u_k^k lie in C_k since the two points l^k, u^k are distinct from l^i, u^i and thus are contained in $S \setminus \{l^i, u^i\}$. This implies that l_k^i and u_k^i lie in C_k . Since u^i, l^i are not contained in C , this forces $l_i^i, u_i^i \notin C_i$. But l^j and u^j lie in C , so l_i^j, u_i^j lie both in C_i and between l_i^i and u_i^i . Thus the length of the interval C_i is strictly less than $|l_i^i - u_i^i|$. We also have that the length of C_i must be at least $|l_i^j - u_i^j|$, so $|l_i^i - u_i^i| > |l_i^j - u_i^j|$. Repeating the above analysis by carving out $S \setminus \{l^j, u^j\}$ shows that $|l_i^i - u_i^i| < |l_i^j - u_i^j|$, a contradiction. Hence $k \leq d + 1$.

With this result, $d + 1 \geq k$ and $k \geq 2n$, implying that $(d + 1)/2 \geq n$. Hence $|S| = d + n \leq (3d + 1)/2$, and since $|S|$ is an integer, $|S| \leq \lfloor (3d + 1)/2 \rfloor$.

4. Attaining the upper bound on the VC dimension

We will show that $VC(\ell^\infty(d))$ is at least $\lfloor (3d + 1)/2 \rfloor$ by recursively constructing a sequence of sets in \mathbb{R}^d that may be shattered by cubes. Some preliminary definitions are required.

Let $d \geq 1$. For any real x , let $R(x) = [x, \infty)$ and $L(x) = (-\infty, x]$. Define

$$H = \{L(x) : x \in \mathbb{R}\} \cup \{R(x) : x \in \mathbb{R}\} \cup \{\mathbb{R}\}.$$

Let $H^d = \{I_1 \times \cdots \times I_d : I_j \in H\}$. If $A \subset \mathbb{R}^d$ is bounded, let

$$\text{diam}(A) = \sup\{\|x - y\|_\infty : x, y \in A\}.$$

Note that for a cube C , $\text{diam}(C)$ is the length of any of its sides, and for a compact subset A of \mathbb{R}^d its diameter is

$$\text{diam}(A) = \max\{|x_i - y_i| : x, y \in A, 1 \leq i \leq d\}.$$

Finally, define $m_j(A) = \min\{a_j : a \in A\}$ and $M_j(A) = \max\{a_j : a \in A\}$, and let the set

$$\text{rect}(A) = [m_1(A), M_1(A)] \times \cdots \times [m_d(A), M_d(A)]$$

be the *rectangular hull* of A .

Lemma 1. *Suppose $d \geq 1$. If $S \subset \mathbb{R}^d$ is finite and $A \subset S$ is carved out by an $I \in H^d$ then there is a cube of diameter $\text{diam}(A)$ contained in I carving out A .*

PROOF. Let $\text{diam}(A) = r$. For every integer j between 1 and d , define an interval C_j in \mathbb{R} as follows. If $I_j = R(x)$ for some real x let C_j be the interval $[m_j(A), m_j(A) + r]$. Otherwise if $I_j = \mathbb{R}$ or $I_j = L(x)$ for some real x , let $C_j = [M_j(A) - r, M_j(A)]$. Note that in both cases C_j is a subset of I_j since both $m_j(A)$ and $M_j(A)$ are assumed to lie in I_j .

Now let $C = C_1 \times \cdots \times C_d$. We have that $C \subset I$ and the diameter of C is r . If x is a point of S not in A , then $x \notin I$ by assumption, giving $x \notin C$. If x is a point in A , then for all j such that $1 \leq j \leq d$, the j th coordinate of x lies in $[m_j(A), M_j(A)]$. Since the length of each C_j is at least the length of this interval and has the endpoint $m_j(A)$ or $M_j(A)$ as defined above, it must be that $x_j \in C_j$. Thus $x \in C$, and so $S \cap C = A$ and $C \subset I$ as desired.

Presented below is the definition of “niceness” that the recursion will require. Note that sets with this property have cardinality one or two less than the claimed VC dimension of cubes. In the proof of the existence of a sequence satisfying these properties, sets of the appropriate cardinality will be constructed that are shattered by cubes. These sets will not have the properties required to continue the recursion.

Definition 2. Let $d \geq 2$ and $S \subset \mathbb{R}^d$. Then we will say that S is nice or shatters nicely if it satisfies the following properties:

1. For every element s of S , $s_1 = -s_2$.
2. The origin lies in $\text{rect}(S)$.
3. (a) If d is even then S has cardinality $\lfloor (3d + 1)/2 \rfloor - 1$. Further, for every subset A of S there is an I in H_d such that I carves out A , the origin lies in I and both I_1 and I_2 are left semi-infinite.
- (b) Otherwise if d is odd then S has cardinality $\lfloor (3d + 1)/2 \rfloor - 2$. In addition every subset A of S may be carved out by an I in H_d such that the origin lies in I and either $I_2 = \mathbb{R}$ and I_1 and I_3 are left semi-infinite, or $I_3 = \mathbb{R}$ and I_1 or I_2 are left semi-infinite.

A final definition and technical lemma is required before the proof itself. For each $d \geq 2$ define a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ such that

$$f(x_1, x_2, \dots, x_d) = (x_1, -x_1, x_2, \dots, x_d)$$

This function will bring the nicely shattered sets S in \mathbb{R}^d to \mathbb{R}^{d+1} with the help of the following lemma.

Lemma 2. *Suppose that $d \geq 2$. Let S be a finite subset of \mathbb{R}^d such that each s in S satisfies $s_1 = -s_2$. If $A \subset S$ can be carved out by an I in H^d then $f(A)$ may be carved out from $f(S)$ by both $I' = I_1 \times \mathbb{R} \times I_2 \times \cdots \times I_d$ and $I'' = I_1 \times I_2 \times \mathbb{R} \times I_3 \times \cdots \times I_d$.*

PROOF. Note immediately that for every $s' \in f(S)$ we have that $s'_2 = s'_3$ by the definition of f and the condition on the points of S . Consider a point $y \in f(S)$. Then there is an x in S such that $f(x) = y$. If $y \in f(A)$ then x lies in A , and so for all j between 1 and d , $x_j \in I_j$. But since $y = (x_1, x_2, x_2, x_3, \dots, x_d)$, $y \in I'$ and $y \in I''$ still.

Otherwise if $y \notin f(A)$, then $x \notin A$, so there is a j such that $x_j \notin I_j$. If $j \neq 2$ then either y_1 is not in I'_1 and I''_1 , or y_{j+1} is not in I'_{j+1} and I''_{j+1} . Otherwise $j = 2$, and so y_3 , which is equal to x_2 , lies neither in I'_3 nor I''_2 . Hence in either case y is not in I' or I'' . Thus $f(S) \cap I' = f(S) \cap I'' = f(A)$.

The proof of the fact that sets that shatter nicely exist for all dimensions at least two will be split over the next three lemmas for greater clarity. We proceed by recursion, establishing the base of $d = 2$ and then showing the existence of a nicely shattering set in an even dimension implies one exists in a dimension higher, and similarly for odd to even dimension.

Lemma 3. *There exists a set S in \mathbb{R}^2 that shatters nicely.*

PROOF. In \mathbb{R}^2 define the two points $b = (1, -1)$ and $c = (-1, 1)$. Let $S = \{b, c\}$. Then $|S| = \lfloor \frac{3(2)+1}{2} \rfloor - 1 = 2$ as required. The condition on the coordinates of b and c is also satisfied, and the origin lies in $\text{rect}(S)$, which is $[-1, 1] \times [-1, 1]$.

It may be easily verified that the four subsets of S

$$\emptyset, \{b\}, \{c\}, \{b, c\}$$

are shattered by the following elements of H_2 respectively:

$$L(0) \times L(0), L(1) \times L(0), L(0) \times L(1), L(1) \times L(1)$$

Each of these elements contain the origin and the first two intervals of each are left semi-infinite. Hence S shatters nicely.

Lemma 4. *Suppose that $d \geq 2$ is even and there exists a set S in \mathbb{R}^d that shatters nicely. Then there exists a set T in \mathbb{R}^d of cardinality $\lfloor (3d+1)/2 \rfloor$ that may be shattered by cubes, and there exists a set T' in \mathbb{R}^{d+1} that shatters nicely.*

PROOF. We will adjoin a single point to S to form the required set T . Define c in \mathbb{R}^d such that $c_1 < m_1(S) - \text{diam}(S)$, $c_2 < m_2(S)$ and $c_i = 0$ for all i at least 3. Let $T = S \cup \{c\}$. Then the set T is shattered by cubes. To verify this, let A be a subset of S and note that every subset of T is of the form A or $A \cup \{c\}$. The subset A may be nicely carved out from S by an I in H_d by assumption. Since I_1 and I_2 are left semi-infinite and I contains the origin, c lies in I . Thus by Lemma 1 there is a cube carving out $A \cup \{c\}$ from S' . As for A itself, note that there is a cube C carving it out from S with diameter $\text{diam}(A)$. This cube, obtained by Lemma 1 has the interval $[M_1(A) - \text{diam}(A), M_1(A)]$. Noting that

$$c_1 < m_1(S) - \text{diam}(S) \leq M_1(A) - \text{diam}(A)$$

the point c does not lie in C , implying $C \cap T = A$. So there is a cube carving out A from T . Hence the set T in \mathbb{R}^d of cardinality $\lfloor (3d+1)/2 \rfloor$ may be shattered by cubes.

Now let $T' = f(T)$ be in \mathbb{R}^{d+1} . By the definition of the function f every t' in T' satisfies $t'_1 = -t'_2$, and the origin lies in $\text{rect}(T')$ still. The set T' also has the required cardinality, as $|T'| = |T| = \lfloor (3d+1)/2 \rfloor = \lfloor (3(d+1)+1)/2 \rfloor - 2$. To verify that the subsets of T' may be shattered by elements of H_d in the appropriate way, let A be a subset of S and $I \in H^d$ nicely carve out A . Also let $A' = f(A)$ be the corresponding subset of T' . Letting $c' = f(c)$, every subset of T' is either of the form $A' \cup \{c'\}$ or A' . In the former case, by Lemma 2 the set A' is carved out from $f(S)$ by $I' = I_1 \times \mathbb{R} \times I_2 \times \cdots \times I_d$. Since $c' = f(c) = (c_1, -c_1, c_2, \dots, c_d)$, the point c' lies in I' . Thus I' carves out $A' \cup \{c'\}$, the set I' still contains the origin, and I' satisfies the appropriate conditions on its first three intervals. As for the latter case, again note that by Lemma 2 the set A' is carved out from $f(S)$ by $I'' = I_1 \times I_2 \times \mathbb{R} \times I_3 \times \cdots \times I_d$. By the definition of f we have the inequality

$$M_2(f(S)) = M_2(S) = -m_1(S) < -c_1 = -c'_1$$

Replace the interval I''_2 in I'' with $R(M_2(f(S)))$. Then the second coordinates of every $s' \in f(S)$ still lie in I''_2 while c'_2 does not. Hence I'' carves out A' from T' , still contains the origin and satisfies the conditions on its first three intervals. So T' indeed shatters nicely.

Lemma 5. *Suppose that $d > 2$ is odd and there exists a set S in \mathbb{R}^d that shatters nicely. Then there exists a set T in \mathbb{R}^d of cardinality $\lfloor (3d+1)/2 \rfloor$*

that may be shattered by cubes, and there exists a set T' in \mathbb{R}^{d+1} that shatters nicely.

PROOF. Define two points b and c in \mathbb{R}^d as follows. Set $b_i < m_i(S)$ for $i \leq 3$, and set b_i equal to 0 for $i > 3$ as necessary. Choose c_1 so that

$$c_1 < m_1(S \cup \{b\}) - \text{diam}(S \cup \{b\}).$$

Let c_2 and c_3 be such that $b_2 < c_2 < m_2(S)$ and $b_3 < c_3 < m_3(S)$, and let $c_i = 0$ for $i > 3$. Finally set $T = S \cup \{b, c\}$. To verify that this set T can be shattered by cubes, let $A \subset S$ and the element I of H^d carve out A from S in accordance with Definition 2. So the intervals I_1 , I_2 and I_3 all extend to the left, and one of I_2 or I_3 is equal to \mathbb{R} . The ability to carve out $A \cup \{b, c\}$ with a cube follows immediately from Lemma 1, since I contains the origin and $b_j, c_j \in I_j$ for $j \leq 3$, giving that $b, c \in I$. To carve out $A \cup \{b\}$, note that $A \cup \{b\} \subset I$ still, so there is a cube C of diameter $\text{diam}(A \cup \{b\})$ containing $A \cup \{b\}$, with C_1 having the left endpoint $M_1(A \cup \{b\}) - \text{diam}(A \cup \{b\})$. Then by the choice of c_1 , the point c is not in C , and so C carves out $A \cup \{b\}$. The tasks of carving out A and $A \cup \{c\}$ are similar to each other. Let j be 2 or 3, to match which interval of I is equal to \mathbb{R} . To carve out A , replace I_j in I with $R(m_j(S))$, and to carve out $A \cup \{c\}$ replace I_j with $R(c_j)$. Then I will carve out A and $A \cup \{c\}$ respectively since $b_j < c_j < m_j(S)$, meaning that there are cubes carving out these sets. As in the previous lemma, every subset of T is of one of those four forms, implying that T may be shattered by cubes.

Consider the set $T' = f(T)$ in \mathbb{R}^{d+1} . Let $b' = f(b)$ and $c' = f(c)$, so $T' = f(T) \cup \{b', c'\}$. This set T' has cardinality $\lfloor (3d+1)/2 \rfloor$, which is equal to $\lfloor (3(d+1)+1)/2 \rfloor - 1$ since d is odd, and every t' in T' has that $t'_1 = -t'_2$ by the definition of f . The origin is still contained in $\text{rect}(T')$ as well. Now let $I' = I_1 \times \mathbb{R} \times I_2 \times \cdots \times I_d$ as in Lemma 2. Carving out the sets $f(A)$, $f(A) \cup \{b'\}$ and $f(A) \cup \{b', c'\}$ with elements of H^{d+1} may be accomplished by replacing the interval I'_2 in I' with $L(M_2(f(S)))$, $L(b'_2)$ and $L(c'_2)$ respectively, since

$$c_2 < b_2 < m_2(S)$$

implies that

$$M_2(f(S)) < b'_2 < c'_2.$$

Note that I'_1 and I'_2 are both left semi-infinite intervals, and the origin remains in I' even with these replacements. To carve out $f(A) \cup \{c'\}$, observe that

either I_2 or I_3 is equal to \mathbb{R} , so I'_3 or I'_4 may be replaced by $R(c'_3)$ or $R(c'_4)$ to give $I' \cap T = f(A) \cup \{c'\}$. The interval I'_2 may be replaced with a left semi-infinite interval $L(x)$ for x sufficiently large to ensure I'_1 and I'_2 are both left semi-infinite while preserving the fact that the origin lies in I' . Thus the set T' in \mathbb{R}^{d+1} shatters nicely.

5. A general bound on the VC dimension of norms in \mathbb{R}^2

In this section $B_\lambda(r)$ will denote an open ball of radius λ and centre r according to some fixed norm in \mathbb{R}^2 . The result that the balls of any such norm have VC dimension at most 3 will come from the consideration of a larger class of convex families obtained in the following way. Let C be a convex set in \mathbb{R}^2 . For any real λ greater than zero and vector r in \mathbb{R}^2 let $\lambda C + r = \{\lambda c + r | c \in C\}$ be the set of vectors in C scaled by λ and then translated by r . Define the collection $\mathcal{C} = \{\lambda C + r | \lambda > 0, r \in \mathbb{R}^2\}$ as the convex family generated by C . It should be noted that every set in \mathcal{C} is still convex. In fact these families arise primarily in the case of norms, where they represent the collection of balls according to a norm. We will prove that such collections have VC dimension at most 3.

It will be useful to recall one final fact about \mathbb{R}^2 . Suppose S is a four-point subset of \mathbb{R}^2 such that no point is in the convex hull of the rest. By Radon's theorem [3], there is a partition of S into two non-empty subsets I and J such that $\text{conv}(I) \cap \text{conv}(J) \neq \emptyset$. Both I and J must have two points, so the convex hulls are simply line segments connecting their two points. These lines cannot be collinear (otherwise one point would necessarily be in the convex hull of the rest), so the point of intersection of their convex hulls must be unique. Call such sets I and J the opposing pairs of S .

The result that the VC dimension of convex families is at most 3 follows from the following technical lemma.

Lemma 6. *Suppose that $S = \{a, b, c, d\}$ is a set of four points none of which is in the convex hull of the rest. Let $I = \{a, c\}$ and $J = \{b, d\}$ be the opposing pairs of S , and $\lambda > 0$ and $r \in \mathbb{R}^2$. Then one of the points in $\{b, d, \lambda a + r, \lambda c + r\}$ lies in the convex hull of the set $\{a, c, \lambda b + r, \lambda d + r\}$.*

PROOF. Let p denote the point of intersection of the convex hulls of I and J . Note then that it suffices to prove that one of the points in

$$\{b - p, d - p, \lambda(a - p) + r + \lambda p, \lambda(c - p) + r + \lambda p\}$$

lies in the convex hull of the set

$$\{a - p, c - p, \lambda(b - p) + r + \lambda p, \lambda(d - p) + r + \lambda p\}.$$

Hence we may assume that $p = 0$, so $\text{conv}(I) \cap \text{conv}(J) = \{0\}$. The sets I and J lie on distinct lines, thus in distinct subspaces of \mathbb{R}^2 . None of the points of S are the origin as that would imply that that point lies in the convex hull of the others, so there exist positive real γ and δ such that $c = -\gamma a$ and $d = -\delta b$. So either

$$a \cdot r \geq 0 \text{ or } c \cdot r \geq 0$$

and either

$$b \cdot r \geq 0 \text{ or } d \cdot r \geq 0.$$

By possible relabelling assume that $a \cdot r \geq 0$ and $b \cdot r \geq 0$. According to the observations above, the set $\{a, b\}$ forms a basis of \mathbb{R}^2 , so write $r = \alpha a + \beta b$ for two non-negative real numbers α and β . Let $\Delta = \alpha - \gamma(\beta + \lambda - 1)$. There are two cases depending on the sign of Δ .

1. We have $\Delta \geq 0$. In this case

$$\lambda c + r = sc + t(\lambda b + r) + u(\lambda d + r),$$

where

$$\begin{aligned} s &= \frac{\lambda\gamma}{\gamma + \alpha} \\ u &= \frac{\Delta}{(1 + \delta)(\gamma + \alpha)} \\ t &= 1 - s - u \end{aligned}$$

To prove that $\lambda c + r$ lies in the convex hull of $\{a, c, \lambda b + r, \lambda d + r\}$, it suffices to prove that the linear combination above is convex. That is, the equality holds, each of s , t and u are non-negative, and $s + t + u = 1$. The last condition holds by design of course.

Expanding $sc + t(\lambda b + r) + u(\lambda d + r)$ by noting that $c = -\gamma a$ and $d = -\delta b$,

$$\begin{aligned} sc + t(\lambda b + r) + u(\lambda d + r) &= s(-\gamma a) + (1 - s - u)(\lambda b + r) \\ &\quad + u(\lambda(-\delta b) + \alpha a + \beta b) \\ &= a(-\gamma s + \alpha(1 - s - u) + \alpha u) \\ &\quad + b((1 - s - u)(\beta + \lambda) - u\lambda\delta + \beta u) \end{aligned}$$

Reducing the coefficient of a ,

$$-\gamma s + \alpha(1 - s - u) + \alpha u = \alpha - s(\gamma + \alpha) = \alpha - \lambda\gamma.$$

Reducing the coefficient of b ,

$$\begin{aligned} (1 - s - u)(\beta + \lambda) - u\lambda\delta + \beta u &= \beta + \lambda - s(\beta + \lambda) - u\lambda(1 + \delta) \\ &= \beta + \lambda - \frac{\lambda\gamma(\beta + \lambda)}{\gamma + \alpha} - \frac{\lambda\Delta}{\gamma + \alpha} \\ &= \beta + \lambda - \lambda = \beta \end{aligned}$$

Thus

$$sc + t(\lambda b + r) + u(\lambda d + r) = a(\alpha - \lambda\gamma) + b\beta = \lambda c + r$$

as desired.

The coefficient s is non-negative as all of its terms are non-negative, and u is non-negative because Δ is. It remains to check that $t = 1 - s - u$ is non-negative, that is, that $1 \geq s + u$. To see this, we compute

$$\begin{aligned} s + u &= \frac{\lambda\gamma}{\gamma + \alpha} + \frac{\Delta}{(1 + \delta)(\gamma + \alpha)} \\ &= \frac{\lambda\gamma(1 + \delta) + \Delta}{(\gamma + \alpha)(1 + \delta)} \\ &\leq \frac{\lambda\gamma(1 + \delta) + \Delta(1 + \delta)}{(\gamma + \alpha)(1 + \delta)} \\ &= \frac{\lambda\gamma + \alpha - \gamma(\beta + \lambda - 1)}{\lambda + \alpha} \\ &= \frac{\alpha + \gamma - \gamma\beta}{\gamma + \alpha} \\ &= 1 - \frac{\gamma\beta}{\gamma + \alpha} \end{aligned}$$

So $s + u \leq 1$. Hence $\lambda c + r$ lies in the convex hull of $\{a, c, \lambda b + r, \lambda d + r\}$.

2. Otherwise $\Delta < 0$. Then

$$b = s(\lambda b + r) + ta + uc$$

where

$$\begin{aligned} s &= \frac{1}{\beta + \lambda} \\ u &= \frac{\alpha + \beta + \lambda - 1}{(\beta + \lambda)(1 + \gamma)} \\ t &= 1 - s - u \end{aligned}$$

As in the first case we need to check that the claimed equality holds and all of s , t and u are non-negative.

Computing $s(\lambda b + r) + ta + uc$,

$$\begin{aligned} s(\lambda b + r) + ta + uc &= s(\lambda b + \alpha a + \beta b) + ta - u\gamma a \\ &= a(s(\alpha + 1 - s - u(1 + \gamma)) + bs(\lambda + \beta)) \end{aligned}$$

The coefficient of b is 1. Expanding the coefficient of a ,

$$\begin{aligned} s(\alpha - 1) - u(1 + \gamma) + 1 &= \frac{\alpha - 1 - \alpha + 1 - \beta - \lambda}{\beta + \lambda} + 1 \\ &= 0 \end{aligned}$$

Thus $s(\lambda b + r) + ta + uc = b$ indeed.

The coefficient $s \geq 0$ automatically, and $u \geq 0$ if $\alpha + \beta + \lambda - 1 \geq 0$.

This follows from the fact that

$$\alpha/\gamma \leq \beta + \lambda - 1$$

and so

$$\alpha \geq -\frac{\alpha}{\gamma} \geq 1 - \beta - \lambda$$

which implies that $u \geq 0$. Finally,

$$s + u = \frac{\alpha + \beta + \gamma + \lambda}{(\beta + \lambda)(1 + \gamma)}$$

and noting that

$$(\beta + \lambda)(1 + \gamma) = \beta + \lambda + \gamma(\beta + \lambda) \geq \alpha + \beta + \gamma + \lambda$$

we have $s + u \leq 1$, so $t \geq 0$. Hence the combination is convex, so b lies in the convex hull of $\{a, c, \lambda b + r, \lambda d + r\}$.

With this lemma we may proceed to the main theorem of this section. Roughly the lemma allows us to conclude that given a set of four points S in \mathbb{R}^2 , either one point lies in the convex hull of the rest (so the other points cannot be separated from that one) or the opposing pairs of S cannot be separated from each other by members of the same convex family.

Theorem 7. *Let S be a set of four points in \mathbb{R}^2 . Then S cannot be shattered by a family of convex sets $\mathcal{C} = \{\lambda C + r \mid \lambda > 0, r \in \mathbb{R}^2\}$ generated by any convex subset C of \mathbb{R}^2 .*

PROOF. If one of the points of S , say x , lies in the convex hull of the others then the subset $S \setminus \{x\}$ cannot be carved out by any convex set (and hence none of the members of \mathcal{C}). Otherwise say $S = \{a, b, c, d\}$ and $I = \{a, c\}$ and $J = \{b, d\}$ are the opposing pairs. Assume towards a contradiction that there are real $\lambda, \lambda' > 0$ and vectors r, r' in \mathbb{R}^2 such that $\lambda C + r$ carves out I and $\lambda' C + r'$ carves out J . We have that x lies in $\lambda' C + r'$ if and only if $(1/\lambda')x - (r'/\lambda')$ lies in C , which is equivalent to $(\lambda/\lambda')x - (\lambda/\lambda')r' + r$ lying in $\lambda C + r$. For clarity let $\mu = \lambda/\lambda'$ be a positive real and $v = r - (\lambda/\lambda')r'$ a vector in \mathbb{R}^2 . Then the points

$$\{a, c, \mu b + v, \mu d + v\}$$

lie in $\lambda C + r$ and the points

$$\{b, d, \mu a + v, \mu c + v\}$$

do not lie in $\lambda C + r$. But by Lemma 6, one of $\{b, d, \mu a + v, \mu c + v\}$ lies in the convex hull of $\{a, c, \mu b + v, \mu d + v\}$. Thus one of those points lies in $\lambda C + r$ since this set is convex, a contradiction.

Thus no such λ, λ' and no r, r' exist, implying that S cannot be shattered by \mathcal{C} .

6. Norms with infinite VC dimension

In this final section we investigate the higher-dimension analogue of the question in the previous section, namely, if the VC dimension of norms in \mathbb{R}^d is uniformly bounded for d greater than 2. This question has a negative answer for all $d \geq 3$; not only is there no bound on the VC dimension of norms in \mathbb{R}^d for such d , there are norms with infinite VC dimension in these spaces. Such norms will be constructed in this section.

A useful property about norms in \mathbb{R}^d is their equivalence with certain types of convex bodies in \mathbb{R}^d . The definition of a convex body in \mathbb{R}^d is a subset of \mathbb{R}^d that is closed, bounded, convex, and has non-empty interior with respect to the standard Euclidean norm. A symmetric convex body is a convex body C with the property that for every v in C , the point $-v$ also lies in C . Any symmetric convex body C in \mathbb{R}^d arises as the unit ball of a norm on \mathbb{R}^d [3]. So if the induced family $\mathcal{C} = \{\lambda C + r | \lambda > 0, r \in \mathbb{R}^3\}$ of a symmetric convex body has infinite VC dimension, the corresponding norm will also have infinite VC dimension.

The following is a statement of the main result of this section, to be proved after a number of preliminary lemmas.

Theorem 8. *Let $d \geq 2$. If the collection of convex bodies in \mathbb{R}^{d-1} has infinite VC dimension, then there is a norm on \mathbb{R}^d with infinite VC dimension.*

It should be stated that the premise of the theorem is satisfied for $d = 3$ by considering the set of convex polygons, for instance. For d greater than 3, either one can take the set of convex polytopes in \mathbb{R}^{d-1} with non-empty interior or proceed by induction and use the family of balls of the norm in \mathbb{R}^{d-1} given by the theorem. The case of $d = 2$ is false by the results of Section 5. In fact the set of convex bodies in \mathbb{R}^1 is exactly the set of closed and bounded intervals, and this collection has VC dimension 2 as discussed in Section 2.

At this point some terminology must be introduced. Given a point x and a subset Q in \mathbb{R}^d , define the cone passing through Q with apex x as

$$\text{Cone}(Q, x) = \{tq + (1 - t)x | q \in Q, t \geq 0\}.$$

It is useful to note that such cones are closed if Q is closed and bounded, and convex if Q is as well. Given a real number β we can define hyperplanes perpendicular to x and passing through βx by

$$H(x, \beta) = \{v \in \mathbb{R}^d | v \cdot x = \beta\}.$$

First we require certain properties of the operation sending a convex set C to $\lambda C + x$ for $\lambda > 0$ and x in \mathbb{R}^d . These are stated in the following lemma.

Lemma 9. *Let Q be a subset of \mathbb{R}^d , the number λ be positive and x a point of \mathbb{R}^d . Then $\lambda Q + x$ is closed, bounded or convex if Q satisfies those properties, and the interior of this set is exactly $\lambda(\text{int}(Q)) + x$. If Q' is a subset of Q , both are convex and contain the origin, and $0 < \lambda' \leq \lambda$ then $\lambda'Q' \subset \lambda Q$.*

PROOF. Consider three of the first implications, that $\lambda Q + x$ is closed or bounded if Q is, and the interior of $\lambda Q + x$ is $\lambda(\text{int}(Q)) + x$. These follow from a slight generalization of an observation made in Section 5. Namely, if $\epsilon > 0$ and $x \in \mathbb{R}^d$ then

$$\lambda B_\epsilon(y) + x = B_{\lambda\epsilon}(\lambda x + y).$$

In particular, the mapping on \mathbb{R}^d sending $v \mapsto \lambda v + x$ preserves open balls. Hence any open ball bounding Q is sent to an open ball bounding $\lambda Q + x$, points in the interior of Q are mapped exactly to the interior points of $\lambda Q + x$, and the set $\mathbb{R}^d \setminus Q$ (which is open if Q is closed) is sent to an open set $\mathbb{R}^d \setminus (\lambda Q + x)$. The result that $\lambda Q + x$ is convex if Q is discussed in Section 5.

Now suppose that Q' is a subset of Q , both are convex and contain the origin, and $0 < \lambda' \leq \lambda$. If u is a point of $\lambda'Q'$, then $u = \lambda'q'$ for some q' in Q' . Noting that $\lambda'/\lambda < 1$ we have that $(\lambda'/\lambda)q' \in Q' \subset Q$. Hence $\lambda'q' = u \in \lambda Q$, so $\lambda'Q' \subset \lambda Q$.

Lemma 10. *Let $d \geq 2$, and suppose the collection \mathcal{C} of closed, bounded and convex sets in \mathbb{R}^d has infinite VC dimension. Then there is a sequence (P_n) of elements of \mathcal{C} such that $P_{n+1} \subset P_n$ for all n , the origin lies in the interior of each P_n and the convex family generated by the set $\{P_n\}_{n \in \mathbb{N}}$ has infinite VC dimension. Further, for each natural N and $m \neq N$ there exists a real $\gamma_m > 0$ such that $\gamma_m P_m \subset P_N$.*

PROOF. By assumption, for each natural N there is a set S_N of cardinality N in \mathbb{R}^d and a collection \mathcal{C}_N of 2^N elements of \mathcal{C} such that S_N is shattered by \mathcal{C}_N . We may take the union of each such \mathcal{C}_N and enumerate the result to obtain a sequence (Q_n) of elements of \mathcal{C} with infinite VC dimension.

Proceed by recursion to construct the appropriate sequence (P_n) from the sequence (Q_n) . For the base $n = 1$, let $P_1 = Q_1$. Now suppose for some natural n that a nested sequence P_1, \dots, P_n has been chosen such that the origin is contained in the interior of each P_i and $P_i = \lambda_i Q_i + r_i$ for appropriate $\lambda_i > 0$ and $r_i \in \mathbb{R}^d$. Let u be a point of the interior of Q_{n+1} . Let Q' be the set $Q_{n+1} - u$. Then the origin is a point in the interior of Q' . Since Q' is still bounded there is a real $R > 0$ such that $Q' \subset B_R(0)$. There is also an $\epsilon > 0$ such that $B_\epsilon(0) \subset P_n$. Letting $\lambda = \epsilon/R$ and $P_{n+1} = \lambda Q'$ we have by Lemma 9,

$$P_{n+1} = \lambda Q' \subset \lambda B_R(0) = B_\epsilon(0) \subset P_n$$

So the set P_{n+1} is still closed, convex and bounded, the origin is a point in its interior and $P_{n+1} \subset P_n$. We also have that $(1/\lambda)P_{n+1} + (1/\lambda)u = Q_{n+1}$. This completes the recursion.

Observe that the method of ensuring that P_{n+1} lies in P_n remains valid. In particular, if N is a natural number and $m \neq N$ then there is a γ_m such that $\gamma_m P_m \subset P_N$. This proves the second assertion.

Before the proof of Theorem 8 we give a result concerning the interaction between hyperplanes and cones that will be used.

Lemma 11. *Let V be a $(d-1)$ -dimensional subspace of \mathbb{R}^d , and w a vector perpendicular to it. Suppose Q is a subset of V . Then for any real numbers α, β, β' such that $\alpha > \beta$ and $\alpha > \beta'$ we have that the intersection of $H(w, \beta)$ with $\text{Cone}(Q + \beta'w, \alpha w)$ is $\frac{\alpha-\beta}{\alpha-\beta'}Q + \beta w$.*

PROOF. Let u lie in $H(w, \beta) \cap \text{Cone}(Q + \beta'w, \alpha w)$. Then there is a q_0 in Q and a $t_0 \geq 0$ such that $u = t_0(q_0 + \beta'w) + (1 - t_0)\alpha w$. In addition, $u \cdot w = \beta$. Noting that $q_0 \cdot w = 0$,

$$u \cdot w = \beta' t_0 + (1 - t_0)\alpha = \beta.$$

That is, $t_0 = (\alpha - \beta)/(\alpha - \beta')$, and so

$$u = t_0(q_0 + \beta'w) + (1 - t_0)\alpha w = \frac{\alpha - \beta}{\alpha - \beta'}q_0 + \beta w,$$

implying that u lies in $\frac{\alpha-\beta}{\alpha-\beta'}Q + \beta w$. It is clear that every point of that form lies in $H(w, \beta) \cap \text{Cone}(Q + \beta'w, \alpha w)$, giving the claimed equality.

Lemma 12. *Let (P_n) be a decreasing sequence of convex bodies in \mathbb{R}^{d-1} . Consider them as subsets of a $(d-1)$ -dimensional subspace V of \mathbb{R}^d , with normal vector w . Then there are real sequences (α_n) , (β_n) and (λ_n) such that $\lambda_n P_n + \beta_n w$ lies in $\text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w)$ for all natural numbers N and n . In addition (α_n) is strictly decreasing, (β_n) is strictly increasing and $\beta_n < \alpha_n$ always.*

PROOF. We recursively construct these sequences based on the following criteria: given, for some natural N , that the terms of the sequences have

been chosen for all $n \leq N$, there are real numbers α_{N+1} , β_{N+1} and λ_{N+1} such that for every $n \leq N$ the following two conditions hold:

$$\lambda_n P_n + \beta_n w \subset \text{Cone}(\lambda_{N+1} P_{N+1} + \beta_{N+1} w, \alpha_{N+1} w)$$

and

$$\lambda_{N+1} P_{N+1} + \beta_{N+1} w \subset \text{Cone}(\lambda_n P_n + \beta_n w, \alpha_n w)$$

In addition the inequality $\beta_N < \beta_{N+1} < \alpha_{N+1} < \alpha_N$ is satisfied.

For the base $N = 1$, set $\lambda_1 = 1$, $\beta_1 = 0$ and $\alpha_1 = 1$. Now suppose that the first N terms of each sequence have been chosen. Let $\beta_{N+1} = (\alpha_N + \beta_N)/2$. Now we set

$$\lambda_{N+1} = \min_{n \leq N} \left\{ \lambda_n \frac{\alpha_n - \beta_{N+1}}{\alpha_n - \beta_n} \right\}$$

This coefficient λ_{N+1} is positive since $\alpha_n > \alpha_N > \beta_{N+1} > \beta_n$ for all $n \leq N$. Then by Lemma 9 we have for each $n \leq N$ that

$$\lambda_{N+1} P_{N+1} \subset \lambda_n \frac{\alpha_n - \beta_{N+1}}{\alpha_n - \beta_n} P_n$$

Hence $\lambda_{N+1} P_{N+1} + \beta_{N+1} w$ is a subset of $\lambda_n \frac{\alpha_n - \beta_{N+1}}{\alpha_n - \beta_n} P_n + \beta_{N+1} w$ and so by Lemma 11, $\lambda_{N+1} P_{N+1} + \beta_{N+1} w$ is a subset of $\text{Cone}(\lambda_n P_n + \beta_n w, \alpha_n w)$. As mentioned in Lemma 10, for each $n \neq N + 1$ there is a positive γ_n such that $\gamma_n P_n \subset P_{N+1}$. Our aim is to choose an α_{N+1} such that $\beta_{N+1} < \alpha_{N+1} < \alpha_N$ and for each $n \leq N$

$$(\lambda_n / \gamma_n) \leq \lambda_{N+1} \frac{\alpha_{N+1} - \beta_n}{\alpha_{N+1} - \beta_{N+1}}.$$

This is possible; by rearrangement the inequality becomes

$$\lambda_n / \gamma_n \leq \lambda_{N+1} \left(1 + \frac{\beta_{N+1} - \beta_n}{\alpha_{N+1} - \beta_{N+1}} \right).$$

Since λ_{N+1} and $\beta_{N+1} - \beta_n$ are both positive, if α_{N+1} is chosen sufficiently close to β_{N+1} then the inequality will be satisfied for all $n \leq N$ while having $\beta_{N+1} < \alpha_{N+1} < \alpha_N$. With this α_{N+1} so defined, we note by Lemma 9 that for each $n \leq N$,

$$\lambda_n P_n = (\lambda_n / \gamma_n) \gamma_n P_n \subset \left(\lambda_{N+1} \frac{\alpha_{N+1} - \beta_n}{\alpha_{N+1} - \beta_{N+1}} \right) P_{N+1}.$$

This implies that $\lambda_n P_n + \beta_n w$ lies in $\text{Cone}(\lambda_{N+1} P_{N+1} + \beta_{N+1} w, \alpha_{N+1} w)$ by Lemma 11.

Noting the equivalence of closed unit balls of some norm and the convex family induced by a symmetric convex body, the proof of an equivalent proposition to Theorem 8 may now be given.

Theorem 13. *Let $d \geq 3$. If the collection of convex bodies in \mathbb{R}^{d-1} has infinite VC dimension, then there is a symmetric convex body F in \mathbb{R}^d whose convex family has infinite VC dimension.*

PROOF. From Lemma 10 there is a nested sequence of convex bodies (P_n) in \mathbb{R}^{d-1} whose convex family has infinite VC dimension. It will be convenient to increment the index of each P_n by 1 (so that P_1 becomes P_2 , and so on) and assume that P_1 is some symmetric convex body containing the rest. As in Lemma 11, regard this sequence as lying in a $(d-1)$ -dimensional subspace V of \mathbb{R}^d with normal vector w . By Lemma 11 there are sequences (α_n) , (β_n) and (λ_n) satisfying the conditions on the cones given in that lemma. As in the proof of that lemma, assume that $\lambda_1 = 1$, $\beta_1 = 0$ and $\alpha_1 = 1$. Define a subset E of \mathbb{R}^d as

$$E = \bigcup_{n \in \mathbb{N}} \lambda_n P_n + \beta_n w.$$

The set E remains bounded. By the conditions on the sequences (α_n) , (β_n) and (λ_n) we have for any two natural numbers N and n ,

$$\lambda_n P_n + \beta_n w \subset \text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w)$$

and hence that

$$E \subset \text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w)$$

for each natural N . Let D be the convex hull of the closure of E . Then D is closed, convex and bounded. Since the set $\lambda_n P_n + \beta_n w$ is closed, bounded and convex for each n , the corresponding $\text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w)$ is closed and convex. Thus

$$D \subset \text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w)$$

for every natural N still. Additionally, note that

$$\lambda_N P_N + \beta_N w \subset E \cap H(w, \beta_N) \subset D \cap H(w, \beta_N),$$

and

$$D \cap H(w, \beta_N) \subset \text{Cone}(\lambda_N P_N + \beta_N w, \alpha_N w) \cap H(w, \beta_N) = \lambda_N P_N + \beta_N w.$$

Hence

$$D \cap H(w, \beta_N) = \lambda_N P_N + \beta_N w.$$

Since D contains the set $\{p+tw | p \in P_2, t \in [0, \beta_2]\}$, it has non-empty interior.

Now let $F = D \cup (-D)$. This set is closed, bounded, has non-empty interior and is symmetric about the origin. It remains to verify that F is convex, making it a symmetric convex body. Since P_1 is symmetric about the origin and is contained in D we have $P_1 \subset -D$. Moreover, $D \setminus P_1$ is the set $D \cap \{v \in \mathbb{R}^d | v \cdot w > 0\}$, so we can write F as the disjoint union

$$F = P_1 \cup (D \setminus P_1) \cup -(D \setminus P_1).$$

Let x and y be two points in F . If they both lie in D or in $-D$ then the convexity of those sets implies that $tx + (1-t)y$ lies in F for any $t \in [0, 1]$. Otherwise suppose that x is in D and y is not in D . Observe that $x \cdot w \geq 0$ and x is contained in the cone $\text{Cone}(P_1, w)$, so for some $t_0 \in [0, 1]$ and p_0 in P_1 ,

$$x = p_0 t_0 + (1 - t_0)w.$$

A similar argument shows that for some $s_0 \in [0, 1)$ and q_0 in P_1 ,

$$y = q_0 s_0 + (1 - s_0)(-w).$$

Then one point in the line segment between x and y , corresponding to a convex combination with coefficients $u = (1 - s_0)/(2 - s_0 - t_0)$ and $1 - u$ is

$$ux + (1 - u)y = \frac{1 - s_0}{2 - s_0 - t_0}(t_0 p_0) + \frac{1 - t_0}{2 - s_0 - t_0}(s_0 q_0)$$

Since t_0 and s_0 do not exceed 1 and P_1 contains the origin, the points $t_0 p_0$ and $s_0 q_0$ lie in P_1 . Thus $ux + (1 - u)y$ itself lies in P_1 . By the convexity of D , the line segment from x to $ux + (1 - u)y$ lies in D , and the line segment from y to $ux + (1 - u)y$ lies in $-D$. Hence the line segment between x and y lies wholly in F . So F remains convex.

The set F is the required symmetric convex body. To shatter a set of cardinality N for any natural number N , we consider the set S in V that was shattered by the convex family of the original sequence (P_n) . If A is a subset of S , then let m be such that $\mu P_m + r$ carves out A for some positive μ and $r \in V$. Then if we let $F' = F - \beta_m w$, the set F' satisfies

$$F' \cap H(w, 0) = F' \cap V = \lambda_m P_m.$$

So $(1/\lambda_m)F' \cap V = P_m$, and then F' may be expanded by μ and translated by r in order to carve out A . Hence the convex family generated by F has infinite VC dimension.

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